On Integrability and Pseudo-Hermitian Systems with Spin-Coupling Point Interactions

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Abstract
We study the pseudo-Hermitian systems with general spin-coupling point interactions and give a systematic description of the corresponding boundary conditions for PT-symmetric systems. The corresponding integrability for both bosonic and fermionic many-body systems with PT-symmetric contact interactions is investigated.

Key words: Point interactions, PT-symmetry, Integrability
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Self-adjoint quantum mechanical models describing a particle moving in a local singular potential have been extensively discussed [1, 2, 3]. The integrability of one dimensional many-body systems with self-adjoint contact interactions has been studied according to Yang-Baxter relations [4]. The results are generalized to the case of particles with spin-coupling interactions [5].

PT-symmetric quantum mechanical models have been studied from some mathematical and physical considerations [6]. In [7] the classification and spectra problem of PT-symmetric point interactions are investigated. The integrability of many-body systems with PT-symmetric interactions is clarified [8]. The δ-type spin-coupling interactions with PT-symmetry is discussed in [9].

In this letter we study the boundary conditions for PT-symmetric point interactions of particles with spin-coupling, and the integrability of bosonic and fermionic many-body systems with PT-symmetric, spin-coupling contact interactions characterized by these boundary conditions.

One dimensional quantum mechanical models of spinless particles with point interactions at the origin can be characterized by separated or nonseparated boundary conditions imposed on the (scalar) wave function φ at $x = 0$. The family of point interactions for the Schrödinger operator $-\frac{d^2}{dx^2}$ can be described by unitary 2×2 matrices via von Neumann formulas for self-adjoint extensions of symmetric operators:

$$\left(\begin{array}{c} \varphi \\ \varphi' \end{array} \right)_{0^+} = e^{i\theta} \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \left(\begin{array}{c} \varphi \\ \varphi' \end{array} \right)_{0^-}, \tag{1}$$

where $ad-bc=1$, $\theta, a, b, c, d \in I\!R$. $\varphi(x)$ is the scalar wave function of two spinless particles with relative coordinate x. (1) also describes two particles with spin s but without any spin coupling between the particles when they meet, in this case φ represents any one of the components of the wave function.

The separated boundary conditions with respect to self-adjoint interactions are described by

$$\varphi'(0_+) = h^+\varphi(0_+)\ , \quad \varphi'(0_-) = h^-\varphi(0_-), \tag{2}$$

where $h^{\pm} \in I\!R \cup \{\infty\}$. $h^+ = \infty$ or $h^- = \infty$ correspond to Dirichlet boundary conditions and $h^+ = 0$ or $h^- = 0$ correspond to Neumann boundary conditions.

The family of PT-symmetric point interactions is described by the boundary conditions at the origin of one of the following two types

$$\begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}_{0_+} = e^{i\theta} \begin{pmatrix} \sqrt{1+bc}\, e^{i\phi} & b \\ c & \sqrt{1+bc}\, e^{-i\phi} \end{pmatrix} \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}_{0_-}; \tag{3}$$

with the real parameters $b \geq 0, c \geq -1/b$ (if the parameter b is equal to zero, then the last inequality can be neglected), $\theta, \phi \in [0, 2\pi)$; and

$$h_0\varphi'(0_+) = h_1 e^{i\theta}\varphi(0_+)\,, \quad h_0\varphi'(0_-) = -h_1 e^{-i\theta}\varphi(0_-) \tag{4}$$

with the real phase parameter $\theta \in [0, 2\pi)$ and with the parameter $\mathbf{h} = (h_0, h_1)$ taken from the (real) projective space $\mathbf{P}^1$.

For a particle with spin s, the wave function has $n = 2s+1$ components. Therefore two particles with contact interactions have a general boundary condition described in the center of mass coordinate system by:

$$\begin{pmatrix} \psi \\ \psi' \end{pmatrix}_{0^+} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \psi \\ \psi' \end{pmatrix}_{0^-}, \tag{5}$$

where ψ and ψ' are n^2-dimensional column vectors, A, B, C and D are $n^2 \times n^2$ matrices. The boundary condition (5) includes not only the usual contact interaction between the particles, but also the spin couplings of the two particles.

For self-adjoint point interactions, due to the required symmetry condition of the Schrödinger operator:

$$< -\frac{d^2}{dx^2}u, v >_{L_2(I\!R,\mathbb{C}^n)} = < u, -\frac{d^2}{dx^2}v >_{L_2(I\!R,\mathbb{C}^n)},$$

for any $u, v \in C^{\infty}(I\!R \setminus \{0\})$, the matrices A, B, C, and D are subject to the following restrictions:

$$A^{\dagger}D - C^{\dagger}B = I\!I, \quad B^{\dagger}D = D^{\dagger}B, \quad A^{\dagger}C = C^{\dagger}A, \tag{6}$$

where $\dagger$ stands for the conjugate and transpose.

Corresponding to (2) the separated boundary conditions are given by

$$\psi'(0_+) = G^+\psi(0_+), \quad \psi'(0_-) = G^-\psi(0_-), \tag{7}$$

where, for the self-adjoint point interactions, $G^\pm$ are Hermitian matrices.

We consider now the boundary conditions describing the point interactions with PT-symmetric and spin-coupling interactions. By applying PT-operation to (5), we have

$$\begin{pmatrix} \psi \\ \psi' \end{pmatrix}_{0^-} = \begin{pmatrix} A^* & -B^* \\ -C^* & D^* \end{pmatrix} \begin{pmatrix} \psi \\ \psi' \end{pmatrix}_{0^+},$$

where $*$ stands for conjugation. Hence A, B, C, D satisfy

$$AA^* - BC^* = I\!I, \quad DD^* - CB^* = I\!I, \quad BD^* = AB^*, \quad CA^* = DC^*, \tag{8}$$

where $I\!I$ is the $n^2 \times n^2$ identity matrix. The boundary condition (5) with A, B, C, D satisfying (8) represents PT-symmetric point interactions with spin coupling.

Accordingly the separated type boundary conditions with respect to PT-symmetric interactions are given by

$$\psi'(0_+) = F\psi(0_+), \quad \psi'(0_-) = G\psi(0_-), \tag{9}$$

where $G = -F^*$.

The case $A = D = I\!I$, $B = 0$, $C = C^*$ corresponds to a Hamiltonian with PT-symmetric δ-type interactions (when C is further symmetric, the system is both PT-symmetric and self-adjoint). The case $A = D = I\!I$, $C = 0$, $B = B^*$ corresponds to a PT-symmetric Hamiltonian H of the form:

$$H = -D_x^2(1 + B\delta) - BD_x\delta',$$

where D_x is defined by $(D_x f)(\varphi) = -f(\frac{d}{dx}\varphi)$, for $f \in C_0^\infty(I\!R/\{0\})$ and φ a test function with a possible discontinuity at the origin. H is self-adjoint when B is Hermitian [5], and H is both self-adjoint and PT-symmetric when B is real symmetric.

Concerning the integrability of many-body systems with spin-coupling interactions, we first consider two-particle case. Let e_α, $\alpha = 1, ..., n$, be the basis (column) vector with the α-th component as 1 and the rest components 0. The wave function of the system is of the form

$$\psi = \sum_{\alpha,\beta=1}^{n} \phi_{\alpha\beta}(x_1, x_2) e_\alpha \otimes e_\beta. \tag{10}$$

According to the statistics ψ is symmetric (resp. antisymmetric) under the interchange of the two particles if s is an integer (resp. half integer). Let k_1 and k_2 be the momenta of the two particles. In the region $x_1 < x_2$, in terms of Bethe hypothesis the wave function has the following form

$$\psi = u_{12}e^{i(k_1x_1+k_2x_2)} + u_{21}e^{i(k_2x_1+k_1x_2)} = u_{12}e^{i(K_{12}X-k_{12}x)} + u_{21}e^{i(K_{12}X+k_{12}x)}, \tag{11}$$

where $X = (x_1 + x_2)/2$, $x = x_2 - x_1$ are the coordinates of the center of mass system, $K_{12} = k_1 + k_2$, $k_{12} = (k_1 - k_2)/2$, u_{12} and u_{21} are $n^2 \times 1$ column matrices.

In the region $x_1 > x_2$,

$$\psi = (P^{12} u_{12}) e^{i(K_{12}X + k_{12}x)} + (P^{12} u_{21}) e^{i(K_{12}X - k_{12}x)}, \tag{12}$$

where according to the symmetry or antisymmetry conditions, $P^{12} = p^{12}$ for bosons and $P^{12} = -p^{12}$ for fermions, p^{12} being the operator on the $n^2 \times 1$ column that interchanges the spins of the two particles. Substituting (11) and (12) into the boundary conditions (5), we get

$$\left\{ \begin{array}{l} u_{12} + u_{21} = A\, P^{12}(u_{12} + u_{21}) + ik_{12} B\, P^{12}(u_{12} - u_{21}), \\ ik_{12}(u_{21} - u_{12}) = C\, P^{12}(u_{12} + u_{21}) + ik_{12} D\, P^{12}(u_{12} - u_{21}). \end{array} \right. \tag{13}$$

Eliminating the term $P^{12}u_{21}$ from (13) we obtain the relation

$$u_{21} = Y_{21}^{12} u_{12}\ , \tag{14}$$

where

$$\begin{array}{rcl} Y_{21}^{12} & = & [(A - ik_{12}B)^{-1} - ik_{12}(C - ik_{12}D)^{-1}]^{-1} \\ & & [(A - ik_{12}B)^{-1}(A + ik_{12}B)P^{12} - (C - ik_{12}D)^{-1}(C + ik_{12}D)P^{12} \\ & & -(A - ik_{12}B)^{-1} - ik_{12}(C - ik_{12}D)^{-1}]. \end{array} \tag{15}$$

Similarly, with respect to the separated type boundary condition (9), we have

$$\begin{array}{l} ik_{12}(u_{21} - u_{12}) = F(u_{12} + u_{21}), \\ ik_{12}P^{12}(u_{12} - u_{21}) = -F^*\, P^{12}(u_{12} + u_{21}). \end{array} \tag{16}$$

The two equations above are compatible when F satisfies

$$(ik_{12} - F)^{-1}(ik_{12} + F) = P^{12}(ik_{12} - F^*)^{-1}(ik_{12} + F^*)P^{12}. \tag{17}$$

(17) is satisfied when F commutes with P^{12}, which implies that F is real. For the case of spin-$\frac{1}{2}$ ($n = 2$), F is generally of the form

$$F = \left(\begin{array}{cccc} a & e_1 & e_1 & c \\ e_3 & f & g & e_2 \\ e_3 & g & f & e_2 \\ d & e_4 & e_4 & b \end{array} \right), \tag{18}$$

where $a, b, c, d, f, g, e_1, e_2, e_3, e_4 \in I\!\!R$. In stead of (15), from (16) we have, upon to the condition (17),

$$Y_{21}^{12} = (ik_{12} - F)^{-1}(ik_{12} + F). \tag{19}$$

In the following we consider the integrability of systems with point interaction described by the boundary condition (9). For a system of N identical particles with PT-symmetric

contact interactions characterized by the separated type operator (19), the wave function in a given region, say $x_1 < x_2 < ... < x_N$, is of the form

$$\begin{aligned} \Psi &= \sum_{\alpha_1,...,\alpha_N=1}^{n} \phi_{\alpha_1,...,\alpha_N}(x_1,...,x_N) e_{\alpha_1} \otimes ... \otimes e_{\alpha_N} \\ &= u_{12...N} e^{i(k_1x_1+k_2x_2+...+k_Nx_N)} + u_{21...N} e^{i(k_2x_1+k_1x_2+...+k_Nx_N)} \\ &\quad +(N!-2) \text{ other terms}, \end{aligned} \tag{20}$$

where k_j, $j = 1, ..., N$, are the momentum of the j-th particle. u are $n^N \times 1$ matrices. The wave functions in the other regions are determined from (20) by the requirement of symmetry (for bosons) or antisymmetry (for fermions). Along any plane $x_i = x_{i+1}$, $i \in 1, 2, ..., N-1$, we have

$$u_{\alpha_1\alpha_2...\alpha_j\alpha_{j+1}...\alpha_N} = Y^{jj+1}_{\alpha_{j+1}\alpha_j} u_{\alpha_1\alpha_2...\alpha_{j+1}\alpha_j...\alpha_N}, \tag{21}$$

where

$$Y^{jj+1}_{\alpha_{j+1}\alpha_j} = [ik_{\alpha_j\alpha_{j+1}} - F_{jj+1}]^{-1}[ik_{\alpha_j\alpha_{j+1}} + F_{jj+1}]. \tag{22}$$

Here $k_{\alpha_j\alpha_{j+1}} = (k_{\alpha_j} - k_{\alpha_{j+1}})/2$ play the role of momenta and $P^{jj+1} = p^{jj+1}$ for bosons and $P^{jj+1} = -p^{jj+1}$ for fermions, where p^{jj+1} is the operator on the $n^N \times 1$ column u that interchanges the spins of particles j and $j+1$. F_{jj+1} stands for the application of the operator F to the jth and $j+1$th particles.

For consistency Y must satisfy the Yang-Baxter equation with spectral parameter [10],

$$Y^{m,m+1}_{ij} Y^{m+1,m+2}_{kj} Y^{m,m+1}_{ki} = Y^{m+1,m+2}_{ki} Y^{m,m+1}_{kj} Y^{m+1,m+2}_{ij}. \tag{23}$$

It is straight forward to verify that the operator Y given by (22) satisfies the Yang-Baxter equation (23). Therefore this system is integrable with the exact wave functions given by (20).

As F is a matrix satisfying (17), it's spectra are not real in general. For instance, the real matrix (18) gives complex eigenvalues. Hence the PT-symmetric (separated type) contact couplings contain non-Hermitian interactions with complex and real spectra, while the real spectra, e.g. spectra of real F commuting with P^{12}, covers part of the real spectra from the Hermitian interactions, when the matrix F is further symmetric ($e_3 = e_1$, $e_4 = e_2$, $d = c$ in the case of (18)).

As for bound states, let us assume that F has real spectra. Let Γ be the set of n^2 eigenvalues of F. For any $\lambda_\alpha \in \Gamma$ such that $\lambda_\alpha < 0$, there are $2^{N(N-1)/2}$ bound states for the N-particle system,

$$\psi^N_{\alpha\underline{\epsilon}} = v_{\alpha\underline{\epsilon}} \prod_{k>l} (\theta(x_k - x_l) + \epsilon_{kl}\theta(x_l - x_k)) e^{\lambda_\alpha \sum_{i>j} |x_i - x_j|}, \tag{24}$$

where $v_{\alpha\underline{\epsilon}}$ is the spin wave function and $\underline{\epsilon} \equiv \{\epsilon_{kl} \,:\, k > l\}$; $\epsilon_{kl} = \pm$, labels the $2^{N(N-1)/2}$-fold degeneracy. The spin wave function v here satisfies $P^{ij} v_{\alpha\underline{\epsilon}} = \epsilon_{ij} v_{\alpha\underline{\epsilon}}$ for any $i \neq j$, that is,

$p^{ij} v_{\alpha\underline{\epsilon}} = \epsilon_{ij} v_{\alpha\underline{\epsilon}}$ for bosons and $p^{ij} v_{\alpha\underline{\epsilon}} = -\epsilon_{ij} v_{\alpha\underline{\epsilon}}$ for fermions. The energy of the bound state $\psi^N_{\alpha\underline{\epsilon}}$ is

$$E_\alpha = -\frac{\lambda_\alpha^2}{3} N(N^2 - 1). \tag{25}$$

We have studied the boundary conditions for systems with general PT-symmetric spin-coupling point interactions, and the corresponding integrability for both bosonic and fermionic many-body systems with separated-type PT-symmetric contact interactions (19). The scattering matrices can be also studied similar to the case of self-adjoint interaction case. The spectra and integrability for many-body systems with PT-symmetric contact interactions described by the Y operator (15) for the non-separated boundary conditions can be studied accordingly in terms of the Yang-Baxter equation (23), though it could be quite complicated as A, B, C, D subject to the conditions (8).

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